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BUCKLING AND POSTBUCKLING OF THE LYING SHEET

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Abstract—The buckling and postbuckling of a finite, heavy elastic sheet is studied. The sheet originally lies on a horizontal plane, one end clamped and one end subjected to a compressive load. Depending on the magnitude of the normalized force F, the lying sheet may be regarded "long" or "short". Stability analyses in both cases show the buckling behavior is quite different from the Euler column theory. A modified critical buckling load is defined. The analytic results compare well with exact numerical integration.

INTRODUCTION

The study of the buckling of a column or an elastic sheet due to a compressive end load is of basic importance in structural engineering. Euler[1] first found the critical buckling load of a weighless column. The standing heavy column subjected to an end load was investigated by Grishcoff[2], and Wang and Drachman[3]. In this paper we shall study the buckling of a heavy column or sheet which lies on a rigid horizontal surface. Problems of this type, having one-sided constraints, are much more difficult than those studied previously using simple stability analysis. As a result, we must revise the usual concept of infinitesimal stability advocated in[2].

FORMULATION

Figure 1(a) shows an elastic sheet of length L and weight per length ρ, one end clamped, lying on a horizontal surface. The other end is subjected to a horizontal compressive force F'. For sufficiently large force F', the elastic sheet will buckle as in Fig. 1(b). Let the origin of a Cartesian axes x', y' be at the free end. Let s' be the arc length from the origin, θ be the local angle of inclination and l' be the lifted length. In Fig. 1(b), only part of the sheet $(0 \leq s' \leq l')$ is lifted. The rest of the sheet $(l' \leq s' \leq L)$ remain flat due to the one-sided constraint of the horizontal surface. If F' is further increased, the entire sheet, except the point at $s' = L$, will separate from the flat surface.

Balancing the local moment m on an elemental length ds', we find

$$dm = \rho s' \cos\theta \, ds' - F' \sin\theta \, ds'. \tag{1}$$

If the sheet is thin enough, the heavy elastica equation of[4] is obtained:

$$EI \frac{d^2\theta}{ds'^2} = \rho s' \cos\theta - F' \sin\theta. \tag{2}$$

We then normalize all lengths by L and drop primes

$$\frac{d^2\theta}{ds^2} = Bs \cos\theta - F \sin\theta. \tag{3}$$

Here $B \equiv \rho L^3/EI$ is a dimensionless parameter representing the relative importance of weight to flexural rigidity EI. The dimensionless parameter $F \equiv F'L^2/EI$ represents the relative importance of compressive load to flexural rigidity. The shape of the sheet is obtained from

$$\frac{dx}{ds} = \cos\theta, \frac{dy}{ds} = \sin\theta. \tag{4}$$

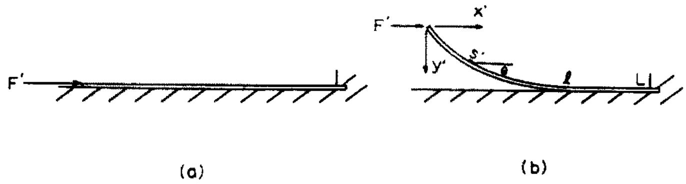

Fig. 1. The lying heavy sheet.

We shall regard a sheet as being *long* if, in a buckled configuration, a segment of the sheet remains flat against the supporting surface, i.e. if $l' < L$. A *short* sheet is one which is completely separated from its horizontal support, except at the attached end, i.e. when $l' = L$. For the long sheet,

$$\frac{d\theta}{ds}(0) = 0, \quad x(0) = y(0) = 0 \tag{5}$$

$$\theta(l) = \frac{d\theta}{ds}(l) = 0. \tag{6}$$

For the short sheet eqn (6) is replaced by

$$\theta(1) = 0. \tag{7}$$

STABILITY OF THE LONG LYING SHEET

Let us consider the case when both θ and B are small. Equation (3) linearized, becomes

$$\frac{d^2\theta}{ds^2} = Bs - F\theta. \tag{8}$$

Notice that eqn (8) applies only to the lifted section $0 \le s \le l \le 1$. Since the trivial solution $\theta = 0$ is *not* a solution to eqn (8), the problem is somewhat different from the usual elastic stability problems. The general solution is

$$\theta = C_1 \sin \sqrt{F}(s - l) + C_2 \cos \sqrt{F}(s - l) + Bs/F. \tag{9}$$

The boundary condition eqn (6) give

$$\theta = -\frac{B}{F\sqrt{F}} \sin \sqrt{F}(s - l) - \frac{Bl}{F} \cos \sqrt{F}(s - l) + Bs/F. \tag{10}$$

If $B \neq 0$, then eqns (5, 10) give the length l:

$$\cos \sqrt{F}l + \sqrt{F}l \sin \sqrt{F}l = 1. \tag{11}$$

The smallest root to eqn (11) is

$$\sqrt{F}l = 2.3311223. \tag{12}$$

In order to buckle infinitesimally, l must start from zero. Thus, it takes infinite force to buckle a horizontal elastic sheet which has nonzero weight! Also, F decreases as the lifted length l increases. This means that for a fixed sufficiently large constant F, there is no infinitesimally stable buckled equilibrium configuration as exists for an Euler column. If the sheet starts to buckle due to imperfection or disturbance, it will continue to collapse until flexural rigidity becomes dominant (the short (*lying*) sheet).

The buckled configuration can be obtained by eqn (4). Retaining the leading terms we have

$$\frac{dx}{ds} \approx 1 - \frac{\theta^2}{2}, \quad \frac{dy}{ds} \approx 0. \tag{13}$$

Using eqn (10), we obtain

$$\begin{aligned} x = s - \frac{B^2}{2}\Bigg\{&\frac{(Fl^2-1)}{4F^3\sqrt{F}}[\sin 2\sqrt{F}(s-l) + \sin 2\sqrt{F}l] \\ &- \frac{l}{2F^3}[\cos 2\sqrt{F}(s-l) - \cos 2\sqrt{F}l] + \frac{2}{F^3}s\cos\sqrt{F}(s-l) \\ &- \frac{2l}{F^3\sqrt{F}}s\sin\sqrt{F}(s-l) - \frac{2}{F^3\sqrt{F}} \\ &\times[\sin\sqrt{F}(s-l) + \sin\sqrt{F}l] - \frac{2l}{F^3}[\cos\sqrt{F}(s-l) \\ &- \cos\sqrt{F}l] + \frac{s^3}{3F^2} + \frac{(1+Fl^2)}{2F^3}s\Bigg\} + 0(B^4) \end{aligned} \tag{14}$$

$$y = B\left[\frac{1}{F^2}\cos\sqrt{F}(s-l) - \frac{l}{F\sqrt{F}}\sin\sqrt{F}(s-l) + \frac{s^2}{2F} - \frac{1}{F^2}\right] + 0(B^3). \tag{15}$$

The horizontal shortening is

$$\begin{aligned} \delta = l - x(l) &= \frac{B^2}{2}\Bigg\{\frac{(Fl^2-1)}{4F^3\sqrt{F}}\sin 2\sqrt{F}l + \frac{l}{2F^3}\cos 2\sqrt{F}l \\ &- \frac{2}{F^3\sqrt{F}}\sin\sqrt{F}l + \frac{2l}{F^3}\cos\sqrt{F}l + \frac{5l^3}{6F^2}\Bigg\} \\ &= 2.3642739\frac{B^2}{F^3\sqrt{F}}. \end{aligned} \tag{16}$$

The maximum vertical displacement is

$$b = y(l) = \frac{Bl^2}{2F} = 2.7170656\frac{B}{F^2}. \tag{17}$$

Although the long lying sheet is unstable for fixed compressive force F, it is stable for given shortening δ. The reaction force can be found from eqns (12) and (16).

The long lying sheet is defined for $l \leq 1$. For small θ, this criterium can be written as

$$F \geq (2.3311223)^2 = 5.434131. \tag{18}$$

Otherwise the elastica is called a short lying sheet.

STABILITY OF THE SHORT LYING SHEET

In this case the weight is unimportant. As first approximation, the sheet behaves like an Euler column. We assume small θ and even smaller parameter B. Set

$$B = \epsilon^3 \ll 1 \tag{19}$$

$$\theta = \epsilon\theta_0 + \epsilon^3\theta_1 + \ldots \tag{20}$$

$$F = F_0 + \epsilon^2 F_1 + \ldots \tag{21}$$

Equation (3) becomes

$$\frac{d^2\theta_0}{ds^2} + F_0\theta_0 = 0. \tag{22}$$

$$\frac{d^2\theta_1}{ds^2} + F_0\theta_1 = s + \frac{F_0\theta_0^3}{6} - F_1\theta_0. \tag{23}$$

The boundary conditions, eqns (5) and (7), give

$$\frac{d\theta_0}{ds}(0) = 0, \quad \theta_0(1) = 0 \tag{24}$$

$$\frac{d\theta_1}{ds}(0) = 0, \quad \theta_1(1) = 0. \tag{25}$$

The zeroth order solution is

$$\theta_0 = C\cos\frac{\pi s}{2}, \quad F_0 = \frac{\pi^2}{4}. \tag{26}$$

Here C is an arbitrary amplitude and F_0 is the Euler buckling load. The effect of weight enters in the first order, eqn (23)

$$\frac{d^2\theta_1}{ds^2} + \frac{\pi^2}{4}\theta_1 = s + \frac{\pi^2}{24}C^3\cos^3\frac{\pi s}{2} - F_1 C\cos\frac{\pi s}{2}. \tag{27}$$

The solution is

$$\theta_1 = C_1\cos\frac{\pi s}{2} - \frac{8}{\pi^3}\sin\frac{\pi s}{2} + \frac{4s}{\pi^2} - \frac{C^3}{192}\cos\frac{3\pi s}{2} + \left(\frac{\pi C^3}{32} - \frac{F_1 C}{\pi}\right)s\sin\frac{\pi s}{2}. \tag{28}$$

C_1 is again an arbitrary constant and can be absorbed into C without loss of generality. Since $\theta_1(1) = 0$, we find

$$F_1 = \frac{\pi^2 C^2}{32} + \frac{1}{C}\left(\frac{4}{\pi} - \frac{8}{\pi^2}\right). \tag{29}$$

For an extremum of F_1, we set $\frac{dF_1}{dC} = 0$ and obtain

$$C = \frac{4}{\pi}\left(1 - \frac{2}{\pi}\right)^{1/3} \tag{30}$$

$$F_1 = \frac{3}{2}\left(1 - \frac{2}{\pi}\right)^{2/3}. \tag{31}$$

Thus the critical buckling force, occurring at finite amplitude C, is

$$\begin{aligned} F_{cr} &= \frac{\pi^2}{4} + \frac{3}{2}\left(1 - \frac{2}{\pi}\right)^{2/3}\epsilon^2 + \dots \\ &= \frac{\pi^2}{4} + \frac{3}{2}\left(1 - \frac{2}{\pi}\right)^{2/3} B^{2/3} + 0(B^{4/3}). \end{aligned} \tag{32}$$

The buckled configuration can be obtained from eqns (4) and the expansion

$$x = x_0 + \epsilon^2 x_1 + \ldots, \tag{33}$$

$$y = \epsilon y_0 + \ldots \tag{34}$$

The solution is

$$x_0 = s, \quad y_0 = \frac{2C}{\pi}\sin\frac{\pi s}{2}, \quad x_1 = -\frac{C^2}{4}\left(s + \frac{1}{\pi}\sin \pi s\right). \tag{35}$$

Thus the horizontal displacement is

$$\delta = 1 - x(1) = \frac{C^2}{4}\epsilon^2 + \ldots = \frac{C^2}{4}B^{2/3} + 0(B^{4/3}). \tag{36}$$

The maximum vertical displacement is

$$b = y(1) = \frac{2C}{\pi}B^{1/3} + 0(B). \tag{37}$$

The moment at $s = 1$ is

$$\frac{\mathrm{d}\theta}{\mathrm{d}s}(1) = -\frac{C\pi}{2}B^{1/3} + \left(\frac{8}{\pi^3} - \frac{C^3\pi}{128}\right)B + 0(B^{5/3}). \tag{38}$$

The amplitude C is related to F by

$$F = \frac{\pi^2}{4} + \left[\frac{\pi^2 C^2}{32} + \frac{1}{C}\left(\frac{4}{\pi} - \frac{8}{\pi^2}\right)\right]B^{2/3} + 0(B^{4/3}). \tag{39}$$

NUMERICAL INTEGRATION

For large deflections the nonlinear governing equations must be integrated numerically. We first transform eqns (3)–(7), which represent a two point boundary value problem, into an initial value problem.

For the long lying sheet, we set

$$s = rB^{-1/3}, \quad x = uB^{-1/3}, \quad y = vB^{-1/3}. \tag{40}$$

eqns (3) and (4) become

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}r^2} = r\cos\theta - H\sin\theta \tag{41}$$

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \cos\theta, \quad \frac{\mathrm{d}v}{\mathrm{d}r} = \sin\theta \tag{42}$$

where $H \equiv FB^{-2/3}$. For given H, we guess $\theta(0)$ and integrate eqns (41) and (42) with the initial conditions

$$\frac{\mathrm{d}\theta}{\mathrm{d}r}(0) = 0, \quad u(0) = 0, \quad v(0) = 0. \tag{43}$$

The integration is terminated when $\mathrm{d}\theta/\mathrm{d}r$ again becomes zero, say, at $r = r^*$. A solution is obtained if θ is also zero there. If not, $\theta(0)$ is adjusted, using Newton's method. The integration is performed on a Textronix computer using the fifth order Runge–Kutta–Fehlberg method with a step size of 0.05.

Then

$$F = HB^{2/3},\quad \delta = [r^* - u(r^*)]B^{-1/3},\quad b = v(r^*)B^{-1/3}. \tag{44}$$

These results are valid for $l < 1$, or $B > (r^*)^3$.

For the short lying sheet, we also use eqns (41)–(43), except that we pick any $\theta(0)$ and integrate until θ reaches zero, say at $r = \bar{r}$. Then

$$B = \bar{r}^3,\quad F = H\bar{r}^2 \tag{45}$$

$$b = v(\bar{r})/\bar{r},\quad \delta = 1 - u(\bar{r})/\bar{r}. \tag{46}$$

RESULTS AND DISCUSSION

Figure 2 shows the vertical edge displacement b as a function of compressive load F for various constant B. Accurate data for the case $B = 0$ (weighless sheet) were from[5]. We see that if gravity is absent, the critical buckling load is the Euler load $\pi^2/4$, and the bifurcation is of the usual pitchfork type. For F greater than the critical load, the straight

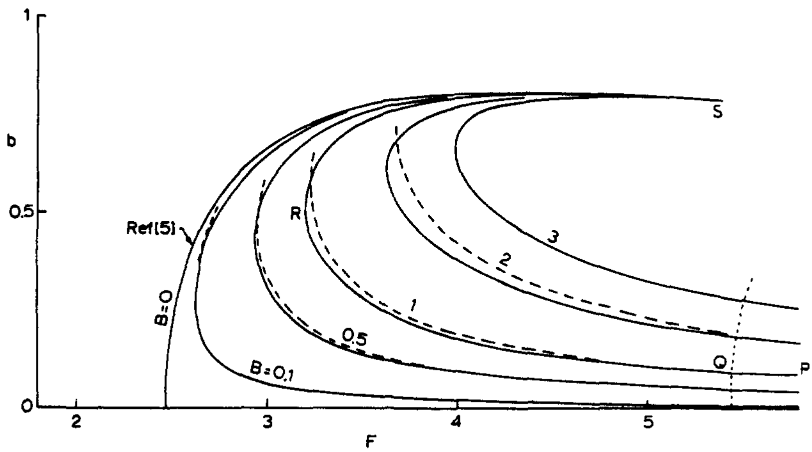

Fig. 2. Vertical edge displacement b as a function of horizontal edge load F. Dotted line separates the long lying sheet from the short lying sheet. Dashed lines are from eqns (37) and (39).

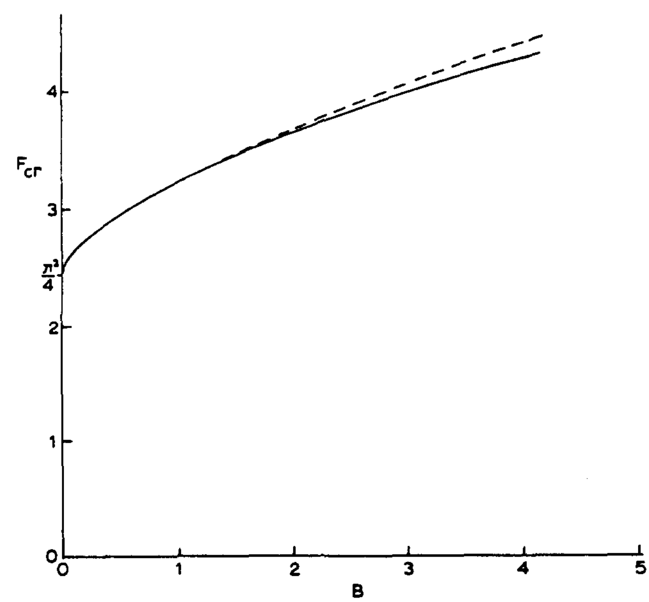

Fig. 3. Critical buckling load as function of B. Dashed line is from eqn (32).

sheet (trivial solution) is unstable to *infinitesimal* disturbances and thus can never be realized.

The situation is quite different if $B \neq 0$. Take for example, the case $B = 1$. The nontrivial equilibrium solution shown on Fig. 2 is the curve $PQRS$, which does not intersect the $b = 0$ axis. The segment QP to the right of the dotted line, represents the long lying sheet. The segment QRS, representing the short lying sheet, approaches the $B = 0$ curve, reflecting the increasing relative effect of flexural rigidity. Our approximate results for small B, eqn (17) for the long lying sheet and eqns (37) and (39) for the short lying sheet, compare well with the exact numerical results.

Now for given finite F, the heavy sheet is stable to infinitesimal disturbances but unstable to *finite* disturbances due to the negative slope of the segment PQR. For example, if $F = 5$ a disturbance of $b > 0.1$ is needed to buckle the sheet which collapses and eventually settle at a state on the stable segment $RS(F = 5, b = 0.795)$. Let us redefine the critical load to be the load below which the sheet will not buckle under *any* disturbance, finite or infinitesimal. This load is at point R, $F_{cr} = 3.2$.

The redefined critical load for the heavy lying sheet is plotted in Fig. 3. The critical load rises sharply from the Euler load $\pi^2/4$ and increases as B is increased. We see that

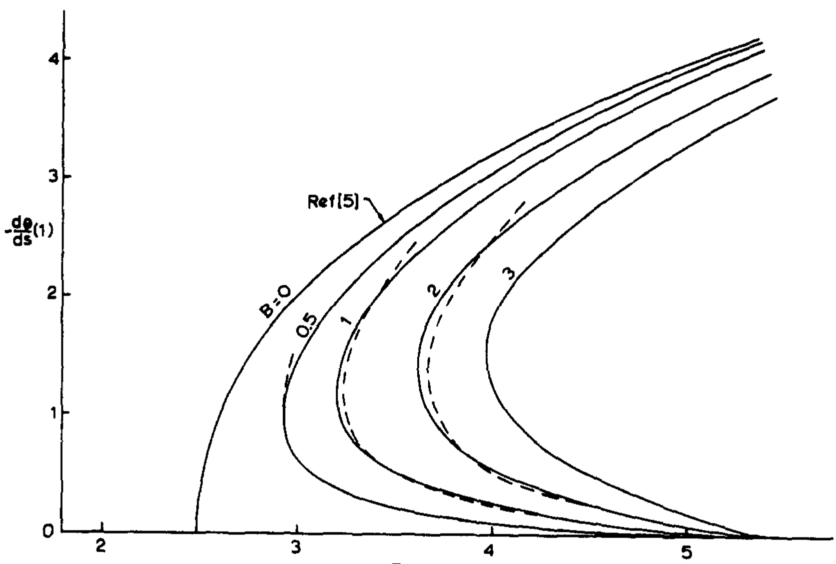

Fig. 4. Moment at the support versus F. Dashed lines are from eqns (38) and (39).

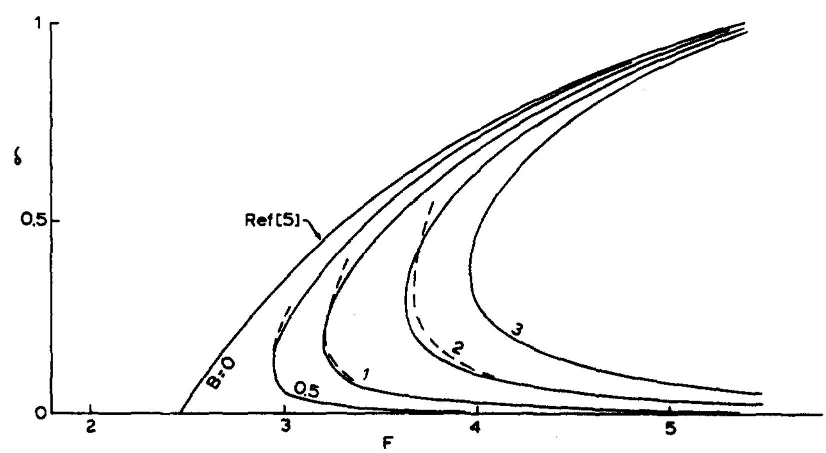

Fig. 5. The horizontal edge displacement versus F. Dashed lines are from eqns (36) and (39).

our approximate solution for small B, given by eqn (32) (dashed curve on Fig. 3), is within 3% error even for larger B values.

Figure 4 shows the moment at the support versus the end load F for various constant values of B. This moment is zero for the long lying sheet, where $F > 5.434$. Figure 5 shows the horizontal edge displacement δ. In general, our analytic approximations are useful for small B and small δ, $\theta'(1)$, or b.

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